

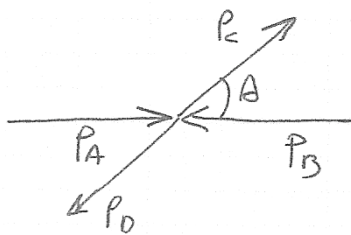
Home Work 5 Solutions

Note: since these solutions are recycled from a previous year, the page numbers are not quite right and the numbering of the problems starts at (2) instead of (1)

② Halzen and Martin 4.3

$$-iM = ie^2 (P_A + P_C)^\mu \frac{1}{q^2} (P_B + P_D)_\mu \quad q = P_C - P_A$$

At high energy, the 4-vectors in the CM frame are



$$P_A = (P \ P \ 0 \ 0)$$

$$P_B = (P \ -P \ 0 \ 0)$$

$$P_C = (P \ p \cos\theta \ p \sin\theta \ 0)$$

$$P_D = (P \ -p \cos\theta \ -p \sin\theta \ 0)$$

$$(P_A + P_C) = \begin{pmatrix} 2P & P(1 + \cos\theta) & P \sin\theta & 0 \end{pmatrix}$$

$$(P_B + P_D) = \begin{pmatrix} 2P & -P(1 + \cos\theta) & -P \sin\theta & 0 \end{pmatrix}$$

$$\begin{aligned} (P_A + P_C)_\mu (P_B + P_D)^\mu &= 4P^2 + P^2(1 + \cos\theta)^2 + P^2 \sin^2\theta \\ &= 4P^2 + P^2 + 2P^2 \cos\theta + P^2 \cos^2\theta + P^2 \sin^2\theta \\ &= 6P^2 + 2P^2 \cos\theta = 2P^2(3 + \cos\theta) \end{aligned}$$

$$q = P_C - P_A = \begin{pmatrix} 0 & P(\cos\theta - 1) & P \sin\theta & 0 \end{pmatrix}$$

$$q^2 = -P^2(\cos\theta - 1)^2 - P^2 \sin^2\theta = -P^2 \cos^2\theta + 2P^2 \cos\theta - P^2 - P^2 \sin^2\theta$$

$$q^2 = 2P^2(\cos\theta - 1)$$

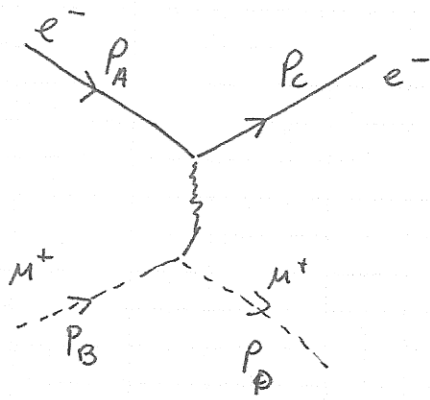
$$\Rightarrow |M|^2 = e^4 \frac{4P^4(3 + \cos\theta)^2}{4P^4(1 - \cos\theta)^2} = e^4 \left(\frac{3 + \cos\theta}{1 - \cos\theta} \right)^2$$

$$\text{Then } \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{P_f}{P_i} |M|^2 = \frac{e^4}{64\pi^2 s} \left(\frac{3 + \cos\theta}{1 - \cos\theta} \right)^2$$

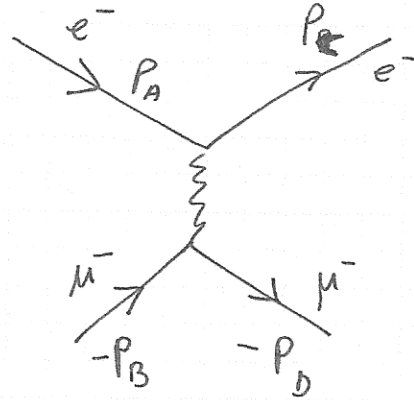
$\swarrow$
 $= 1$

$$\text{But } e^4 = 16\pi^2 \alpha^2 \Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} \left(\frac{3 + \cos\theta}{1 - \cos\theta} \right)^2}$$

③ Helsen & Martin 4.4



$$e^- \mu^+ \rightarrow e^- \mu^+$$



For $e^- \mu^- \rightarrow e^- \mu^-$ (AB $\rightarrow$ CD) we had

$$-iM = ie (p_A + p_C)^\mu \left(\frac{-i g_{\mu\nu}}{q^2} \right) (p_B + p_D)^\nu \quad q = p_A - p_C$$

$$\Rightarrow \boxed{-iM = -ie (p_A + p_C)^\mu \left(\frac{-i g_{\mu\nu}}{q^2} \right) (p_B + p_D)^\nu}$$

Now try crossing

$$e^- \mu^- \rightarrow e^- \mu^-$$

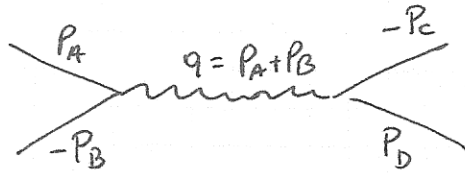
$$A \rightarrow C, B \rightarrow D$$

$$-iM = ie (p_A + p_C)^\mu \left(\frac{-i g_{\mu\nu}}{q^2} \right) (p_B + p_D)^\nu$$

cross $e^- \mu^- \rightarrow e^- \mu^-$
 $A(-D) \rightarrow C(-B) \Rightarrow$ gives same result as

$$e^- e^+ \rightarrow \mu^+ \mu^-$$

$$A B \rightarrow C D$$



$$-iM = +ie (P_A - P_B)^\mu \left(\frac{-i g_{\mu\nu}}{q^2} \right) (P_D - P_C)^\nu$$

Now try crossing

$$e^- \mu^- \rightarrow e^- \mu^-$$

$$A B \rightarrow C D$$

$$\text{was } -iM = +ie (P_A + P_C)^\mu \left(\frac{-i g_{\mu\nu}}{q^2} \right) (P_D + P_B)^\nu$$

Cross: $e^- e^+ \rightarrow \mu^+ \mu^-$

$$A - C \rightarrow -B D$$

$$q = P_A - P_C$$

So take expression for $e^- \mu^- \rightarrow e^- \mu^-$, substitute $P_B \rightarrow -P_C$
 $P_C \rightarrow -P_B$

$$\Rightarrow -iM = +ie (P_A - P_B)^\mu \left(\frac{-i g_{\mu\nu}}{(P_A + P_B)^2} \right) (P_D - P_C)^\nu$$

④ H&M 5.4

$$\begin{aligned}
 [H, L_i] &= [\alpha_e p_e + \beta m \quad \epsilon_{ijk} x_j p_k] \\
 &= \epsilon_{ijk} \alpha_e p_e x_j p_k - \epsilon_{ijk} \alpha_e x_j p_k p_e + \beta m \epsilon_{ijk} x_j p_k - \epsilon_{ijk} \beta m x_j p_k \\
 &= \epsilon_{ijk} \alpha_e [-i \delta_{ej} p_k + x_j p_e p_k] - \epsilon_{ijk} \alpha_e x_j p_k p_e \\
 &= -i \epsilon_{ijk} \alpha_e \delta_{ej} p_k = -i \epsilon_{ijk} \alpha_j p_k = -i [\vec{\alpha} \times \vec{p}]
 \end{aligned}$$

$$[H, \Sigma_j] = [\alpha_i p_i + \beta m \quad \Sigma_j]$$

Look first at $[\beta, \Sigma_j] = \left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix} \right] = 0$

So $[H, \Sigma_j] = [\alpha_i p_i, \Sigma_j] = \begin{pmatrix} 0 & \sigma_j p_i \\ \sigma_j p_i & 0 \end{pmatrix} \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix} - \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix} \begin{pmatrix} 0 & \sigma_j p_i \\ \sigma_j p_i & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & \sigma_i \sigma_j p_i \\ \sigma_i \sigma_j p_i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_j \sigma_i p_i \\ \sigma_j \sigma_i p_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & [\sigma_i, \sigma_j] p_i \\ [\sigma_i, \sigma_j] p_i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2i \epsilon_{ijk} \sigma_k p_i \\ 2i \epsilon_{ijk} \sigma_k p_i & 0 \end{pmatrix} = 2i \epsilon_{ijk} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} p_i$$

$$= 2i \epsilon_{ijk} \alpha_k p_i = 2i \epsilon_{kij} \alpha_k p_i = 2i (\vec{\alpha} \times \vec{p})_j$$

$$\Rightarrow [H, \vec{\Sigma}] = 2i (\vec{\alpha} \times \vec{p})$$

Then $\vec{J} = \vec{L} + \frac{1}{2} \vec{\Sigma}$ is now such that $[H, \vec{J}] = 0$

5 H&M 5.6

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \vec{\gamma} = \vec{\beta} \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \quad \text{Let } A = C \gamma^0 = i \gamma^2$$

• Want to show that $-A \gamma^{\mu*} = \gamma^{\mu} A$

First $\mu=0$: $-A \gamma^{0*} = -A \gamma^0 = -i \gamma^2 \gamma^0 = +i \gamma^0 \gamma^2 = \gamma^0 A \checkmark$

Then $\mu=1,3$ $-A \gamma^{\mu*} = -A \gamma^{\mu} = -i \gamma^2 \gamma^{\mu} = i \gamma^{\mu} \gamma^2 = \gamma^{\mu} A \checkmark$

Then $\mu=2$ $-A \gamma^{2*} = +A \gamma^2 = i \gamma^2 \gamma^2 = \gamma^2 A \checkmark$

• Want to show that $\psi_c^{(1)} = i \gamma^2 \psi_c^{(1)*} = v^{(1)}(\vec{p}) e^{ipx}$

$$i \gamma^2 \psi_c^{(1)*} = i \gamma^2 [u^{(1)}(\vec{p}) e^{-ipx}]^* = i \gamma^2 u^{(1)*} e^{ipx}$$

$$= i \gamma^2 \begin{pmatrix} 1 \\ 0 \\ \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}^* e^{ipx} = i \gamma^2 \begin{bmatrix} 1 \\ 0 \\ \frac{1}{E+m} \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{bmatrix}^* e^{ipx}$$

$$= i \gamma^2 \begin{bmatrix} 1 \\ 0 \\ p_z/E+m \\ p_x + ip_y/E+m \end{bmatrix}^* e^{ipx} = i \gamma^2 \begin{bmatrix} 1 \\ 0 \\ p_z/E+m \\ \frac{p_x - ip_y}{E+m} \end{bmatrix} e^{ipx}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ p_z/E+m \\ \frac{p_x - ip_y}{E+m} \end{bmatrix} e^{ipx} = \begin{bmatrix} \frac{p_x - ip_y}{E+m} \\ -p_z/E+m \\ 0 \\ 1 \end{bmatrix} e^{ipx}$$

$$= \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi^2 \\ \chi^2 \end{pmatrix} e^{ipx} = u^{(4)}(-\vec{p}) e^{ipx} = v^{(1)}(\vec{p}) e^{ipx} \checkmark$$

• Want to show that $C^{-1} \gamma^{\mu} C = (-\gamma^{\mu})^T$

$$C = -C^{-1} = -C^{\dagger} = -C^{tr}$$

$$-(C\gamma^0)\gamma^{\mu*} = \gamma^{\mu}(C\gamma^0)$$

$$\Rightarrow \gamma^{\mu} = -(C\gamma^0)\gamma^{\mu*}(C\gamma^0)^{-1} = -C\gamma^0\gamma^{\mu*}\gamma^0 C^{-1}$$

$$\Rightarrow C^{-1}\gamma^{\mu}C = -\gamma^0\gamma^{\mu*}\gamma^0$$

We now need to show that $-\gamma^0\gamma^{\mu*}\gamma^0 = (\gamma^{\mu})^{Tr}$

Look at $\mu=0$ first $-\gamma^0\gamma^{0*}\gamma^0 = -\gamma^0\gamma^0\gamma^0 = -\gamma^0 = (\gamma^0)^{Tr}$ ✓

Now $\mu=1,3$

$$-\gamma^0\gamma^{\mu*}\gamma^0 = -\gamma^0\gamma^{\mu}\gamma^0 = +\gamma^{\mu} = (\gamma^{\mu})^{Tr}$$

No $\mu=2$

$$-\gamma^0\gamma^{2*}\gamma^0 = \gamma^0\gamma^2\gamma^0 = -\gamma^0\gamma^0\gamma^2 = -\gamma^2$$

But $\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \Rightarrow \gamma^2 = (\gamma^2)^{Tr} \Rightarrow -\gamma^0\gamma^{2*}\gamma^0 = (\gamma^2)^{Tr}$

$$C\gamma^0 = i\gamma^2 \Rightarrow C = i\gamma^2\gamma^0$$

Consider $C^2 = (i\gamma^2\gamma^0)(i\gamma^2\gamma^0) = -\gamma^2\gamma^0\gamma^2\gamma^0 = +\gamma^2\gamma^2\gamma^0\gamma^0 = (-1)(1) = -1$

$$\Rightarrow \boxed{C^{-1} = -C}$$

Consider $C^{\dagger} = (i\gamma^2\gamma^0)^{\dagger} = -i\gamma^{0\dagger}\gamma^{2\dagger} = -i\gamma^0(-\gamma^2) = i\gamma^0\gamma^2 = -i\gamma^2\gamma^0 = -C$

$$\Rightarrow \boxed{C = -C^{\dagger}}$$

Consider $C^{Tr} = (i\gamma^2\gamma^0)^{Tr} = (C^{\dagger})^* = -C^* = -C \Rightarrow \boxed{C^{Tr} = -C}$

• Finally we want to show that $\bar{\psi}_c = -\psi^{Tr} C^{-1}$

$$\begin{aligned} \bar{\psi}_c &= \psi_c^{\dagger}\gamma^0 = (C\gamma^0\psi^*)^{\dagger}\gamma^0 = \psi^{Tr}\gamma^{0\dagger}C^{\dagger}\gamma^0 \\ &= \psi^{Tr}\gamma^0(-C)\gamma^0 = \psi^{Tr}\gamma^0(-i\gamma^2\gamma^0)\gamma^0 \\ &= \psi^{Tr}\gamma^0(-i\gamma^2) = \psi^{Tr}(i\gamma^2\gamma^0) = \psi^{Tr}C = -\psi^{Tr}C^{-1} \end{aligned}$$

6) Holzen & Martin 5.13

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$$P_L + P_R = \frac{1}{2}(1 + \gamma^5) + \frac{1}{2}(1 - \gamma^5) = 1 \quad \checkmark$$

$$P_L P_R = \frac{1}{4}(1 + \gamma^5)(1 - \gamma^5) = \frac{1}{4}(1 - \gamma^5 \gamma^5 + \gamma^5 - \gamma^5) = \frac{1}{4}(1 - \gamma^5 \gamma^5) = 0 \quad \checkmark$$

$$(P_i)^2 = \frac{1}{4}(1 \pm \gamma^5)^2 = \frac{1}{4}(1 + \gamma^5 \gamma^5 \pm 2\gamma^5) = \frac{1}{2}(1 \pm \gamma^5) = P_i \quad \checkmark$$

7) Holzen & Martin 5.14

$$[H, \gamma^5] = [\vec{\alpha} \vec{p} + \beta m, \gamma^5]$$

$$\text{Consider } [\alpha_i, \gamma^5] = [\gamma^0 \gamma^i, \gamma^5] \quad (\text{since } \vec{\gamma} = \beta \vec{\alpha} = \gamma^0 \vec{\alpha} \Rightarrow \vec{\alpha} = \gamma^0 \vec{\gamma})$$

Then

$$\begin{aligned} [\alpha_i, \gamma^5] &= \gamma^0 \gamma^i \gamma^5 - \gamma^5 \gamma^i \gamma^0 = i \gamma^0 \gamma^i \gamma^0 \gamma^1 \gamma^2 \gamma^3 - i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 \gamma^i \\ &= -i \gamma^i \gamma^0 \gamma^0 \gamma^1 \gamma^2 \gamma^3 + i \gamma^1 \gamma^2 \gamma^3 \gamma^0 \gamma^0 \gamma^i \\ &= -i \gamma^i \gamma^1 \gamma^2 \gamma^3 + i \gamma^1 \gamma^2 \gamma^3 \gamma^i \end{aligned}$$

Take $i=1$

$$\begin{aligned} [\alpha_1, \gamma^5] &= -i \gamma^1 \gamma^1 \gamma^2 \gamma^3 + i \gamma^1 \gamma^2 \gamma^3 \gamma^1 = +i \gamma^2 \gamma^3 + \gamma^1 \gamma^1 \gamma^2 \gamma^3 \\ &= +i \gamma^2 \gamma^3 - \gamma^2 \gamma^3 = 0 \end{aligned}$$

Take $i=2$

$$[\alpha_2, \gamma^5] = -i \gamma^2 \gamma^1 \gamma^2 \gamma^3 + i \gamma^1 \gamma^2 \gamma^3 \gamma^2 = +i \gamma^1 \gamma^2 \gamma^2 \gamma^3 - i \gamma^1 \gamma^2 \gamma^2 \gamma^3 = 0$$

Take $i=3$

$$[\alpha_3 \gamma^5] = -i \gamma^3 \gamma^1 \gamma^2 \gamma^3 + i \gamma^1 \gamma^2 \gamma^3 \gamma^3 = -i \gamma^3 \gamma^3 \gamma^1 \gamma^2 - i \gamma^1 \gamma^2 = 0$$

$$\text{So } [\vec{\alpha} \vec{p} \gamma^5] = 0$$

$$\text{Now consider } [\beta m \gamma^5] = m [\gamma^0 \gamma^5] = i m \gamma^0 \gamma^0 \gamma^1 \gamma^2 \gamma^3 - i m \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0$$

$$[\beta m \gamma^5] = i m \gamma^1 \gamma^2 \gamma^3 + i m \gamma^0 \gamma^0 \gamma^1 \gamma^2 \gamma^3 = 2 i m \gamma^1 \gamma^2 \gamma^3$$

$$\Rightarrow [H \gamma^5] = 2 i m \gamma^1 \gamma^2 \gamma^3 \neq 0$$

HANDEDNESS IS NOT A
GOOD QUANTUM NUMBER
UNLESS $m=0$

$$\text{Helicity } \begin{pmatrix} \vec{\sigma} \cdot \vec{p} & 0 \\ 0 & \vec{\sigma} \cdot \vec{p} \end{pmatrix} = \Sigma(\hat{p})$$

$$H = \vec{\alpha} \vec{p} + \beta m = \begin{pmatrix} -\vec{\sigma} \cdot \vec{p} & m \\ m & \vec{\sigma} \cdot \vec{p} \end{pmatrix}$$

$$\begin{aligned} [H \Sigma(\hat{p})] &= \begin{pmatrix} -\vec{\sigma} \cdot \vec{p} & m \\ m & \vec{\sigma} \cdot \vec{p} \end{pmatrix} \begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix} = \begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix} \begin{pmatrix} -\vec{\sigma} \cdot \vec{p} & m \\ m & \vec{\sigma} \cdot \vec{p} \end{pmatrix} \\ &= \begin{pmatrix} -|\vec{p}| & m \vec{\sigma} \cdot \hat{p} \\ m \vec{\sigma} \cdot \hat{p} & |\vec{p}| \end{pmatrix} - \begin{pmatrix} -|\vec{p}| & m \vec{\sigma} \cdot \vec{p} \\ m \vec{\sigma} \cdot \vec{p} & |\vec{p}| \end{pmatrix} = 0 \end{aligned}$$

HELICITY IS CONSERVED

Since helicity is $\Sigma(\hat{p})$, and $\hat{p}$ is frame dependent, helicity is frame dependent. In particular, if we reverse the sign of $\vec{p}$, we reverse the sign of helicity also.

⑧ Holten and Martin 5.15

In Dirac Pauli $\gamma^5 = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}$

$$\gamma^5 u^{(s)} = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} \begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi^{(s)} \end{pmatrix}$$

For $p \gg m$ $\frac{\vec{\sigma} \cdot \vec{p}}{E+m} \rightarrow \frac{\vec{\sigma} \cdot \vec{p}}{E} \approx \frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} = \cancel{\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}} = \vec{\sigma} \cdot \hat{p}$

Then

$$\gamma^5 u^{(s)} \approx \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} \begin{pmatrix} \chi^{(s)} \\ \vec{\sigma} \cdot \hat{p} \chi^{(s)} \end{pmatrix} = \begin{pmatrix} \vec{\sigma} \cdot \hat{p} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$$

$$\gamma^5 u^{(s)} \approx \begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix} \begin{pmatrix} \chi^{(s)} \\ \vec{\sigma} \cdot \hat{p} \chi^{(s)} \end{pmatrix} = \begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix} u^{(s)}$$

(Using the fact that $(\vec{\sigma} \cdot \hat{p})(\vec{\sigma} \cdot \hat{p}) = |\hat{p}|^2 = 1$)